CONVERSE PROBLEM OF ELASTICITY THEORY FOR AN ANISOTROPIC MEDIUM

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To prevent stress concentration it is of considerable interest that the body contour be found whose portions show no preference to brittle failure or plastic deformation. Such contours are called by us "equally rigid." Two-dimensional problems are considered for finding the "equally rigid" form of the hole in an anisotropic medium. The lack of stress concentration on the coutour of the hole is the criterion which decides whether or not the hole is "equally rigid." For an isotropic medium this converse problem of elasticity theory was solved in [1].

The determination of an "equally rigid" contour of the hole is considered in an anisotropic medium, the latter being in a homogeneous stress field,

$$\sigma_x = \sigma_x^\infty,\ \sigma_y = \sigma_y^\infty,\ \tau_{xy} = 0.$$

Suppose that to the unknown contour L of the hole a constant load along the normal and a vanishing tangential load are applied

$$\sigma_n = -p,\ \tau_{tn} = 0 \tag{1}$$

(t and n are the directions of the tangent or the normal to L, respectively).

It is required that the relation

$$\sigma_t = \sigma_* = \text{const} \tag{2}$$

be valid at all points of the unknown contour L. The constant σ_* should also be determined when solving the problem.

The stresses σ_x, σ_y, τ_{xy} in a two-dimensional problem of elasticity theory for an anisotropic body are given by means of two analytic functions (z_1) and $\psi(z_2)$ [2]:

$$\begin{aligned} \sigma_x &= 2\,\mathrm{Re}\left[s_1^2\varphi'(z_1) + s_2^2\psi'(z_2)\right]; \\ \sigma_y &= 2\,\mathrm{Re}\left[\varphi'(z_1) + \psi'(z_2)\right]; \\ \tau_{xy} &= -2\,\mathrm{Re}\left[s_1\varphi'(z_1) + s_2\psi'(z_2)\right], \end{aligned} \tag{3}$$

where

$$\varphi'(z_1) = \frac{d\varphi(z_1)}{dz_1};\quad \psi'(z_2) = \frac{d\psi(z_2)}{dz_2};\quad z_1 = x + s_1 y;$$

$$z_2 = x + s_2 x,\ \text{and}\ s_1 = \alpha_1 + i\beta_1\ \text{and}\ s_2 = \alpha_2 + i\beta_2$$

are the roots of the equation $a_{11}s^4 - 2a_{16}s^3 + (2a_{12} + a_{66})s^2 - 2a_{26}s + a_{22} = 0$ (a_{jk} are elastic constants).

Side by side with the given plane of z = x + iy one also considers the planes of z_1

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and z_2; the latter are obtained from the z plane by means of affine mappings,

$$z_1=x+s_1y=x_1+iy_1;\quad z_2=x+s_2y=x_2+iy_2.$$

The above mappings transform the unknown contour into the contours L_1 and L_2 in the plane z_1 and z_2. If the external forces are (1) and (2), the boundary conditions for the analytic functions $\varphi(z_1)$ and $\psi(z_2)$ can be represented as follows [2, 3]:

$$(1+is_1)\varphi(z_1)+(1+i\bar{s}_1)\overline{\varphi(z_1)}+(1+is_2)\psi(z_2)+(1+i\bar{s}_2)\overline{\psi(z_2)}=-pz+\text{const}; \tag{4}$$

$$\left(1+s_1^2\right)\varphi'(z_1)+\left(1+\bar{s}_1^2\right)\overline{\varphi'(z_1)}+\left(1+s_2^2\right)\psi'(z_2)+\left(1+\bar{s}_2^2\right)\overline{\psi'(z_2)}=\frac{1}{2}(\sigma_*-p). \tag{5}$$

At infinity the functions $\varphi(z_1)$ and $\psi(z_2)$ behave as follows:

$$\varphi(z_1)=B^*z_1+0\left(\frac{1}{z_1}\right);\quad \psi(z_2)=B_1^*z_2+0\left(\frac{1}{z_2}\right),$$

where

$$B^*=\frac{\sigma_x^\infty+\left(\alpha_2^2+\beta_2^2\right)\sigma_y^\infty}{2\left[(\alpha_2-\alpha_1)^2+\left(\beta_2^2-\beta_1^2\right)\right]};\quad B_1^*=B^{1*}+iC^*;$$

$$B^{1*}=\frac{\left(\alpha_1^2-\beta_1^2\right)\sigma_y^\infty-2\alpha_1\alpha_2\sigma_y^\infty-\sigma_x^\infty}{2\left[(\alpha_2-\alpha_1)^2+\left(\beta_2^2-\beta_1^2\right)\right]};$$

$$C^*=\frac{(\alpha_1-\alpha_2)\sigma_x^\infty+\left[\alpha_2\left(\alpha_1^2-\beta_1^2\right)-\alpha_1\left(\alpha_2^2-\beta_2^2\right)\right]\sigma_y^\infty}{2\left[(\alpha_2-\alpha_1)^2-\left(\beta_2^2-\beta_1^2\right)\right]\beta_2}.$$

The constant on the right-hand side of the relation (4) can be set equal to zero.

The change to the parametric plane ξ is carried out by means of the mapping $z=\omega(\zeta)$:

$$\omega(\zeta)=R\left(\zeta+\sum_{k=1}^{\infty}c_k\zeta^{-k}\right).$$

Let the analytic function $\omega(\xi)$ represent the conformal mapping of the outside of the unit circle in the ξ plane into the outside of the contour L in the z plane.

The functions $z_1=\omega_1(\zeta)$ and $z_2=\omega_2(\zeta)$ represent the conformal mappings of the outside of the unit circle of the plane ξ into the outside of the contours L_1 and L_2; to the points M, M_1, and M_2 of the contours L, L_1, and L_2 which are in affine correspondence there corresponds a single point on the circumference of the unit circle. The following notation is now introduced: $\varphi(\zeta)=\varphi[\omega_1(\zeta)]$, $\psi(\zeta)=\psi[\omega_2(\zeta)]$

$$\Phi(\zeta)=\frac{\varphi'(\zeta)}{\omega_1'(\zeta)},\quad \Psi(\zeta)=\frac{\psi'(\zeta)}{\omega_2'(\zeta)}.$$

The boundary conditions (4) and (5) are now used to determine the three analytic functions $\varphi(\zeta)$, $\psi(\zeta)$, and $\omega(\zeta)$; this results in the following nonlinear boundary-value problem at $|\zeta|=1$:

$$(1+is_1)\varphi(\zeta)+(1+i\bar{s}_1)\overline{\varphi(\zeta)}+(1+is_2)\psi(\zeta)+(1+i\bar{s}_2)\overline{\psi(\zeta)}=-p\omega(\zeta); \tag{6}$$

$$\left(1+s_1^2\right)\Phi(\zeta)+\left(1+\bar{s}_1^2\right)\overline{\Phi(\zeta)}+\left(1+s_2^2\right)\Psi(\zeta)+\left(1+\bar{s}_2^2\right)\overline{\Psi(\zeta)}=\frac{1}{2}(\sigma_*-p). \tag{7}$$

Applying the method of functional equations [4] one finds the solutions of the boundary-value problem (6) and (7) given by

$$\varphi(\zeta)=\frac{R}{2}(1+c_1-is_1+is_2c_1)B^*\zeta+\frac{R}{2i(s_2-s_1)\zeta}\times$$
$$\times\{(1-is_2)c_1p-[i(s_2-\bar{s}_1)(1+c_1+i\bar{s}_1-i\bar{s}_1c_1)B^*+$$
$$+i(s_2-\bar{s}_2)(1+c_1+i\bar{s}_2-i\bar{s}_2c_1)B_1^*+p(1+is_2)]\}=a_1\zeta+\frac{a_{-1}}{\zeta}; \tag{8}$$

$$\psi(\zeta)=\frac{R}{2}(1+c_1-is_2+is_2c_1)B^*\zeta+\frac{R}{2i(s_1-s_2)\zeta}\times$$
$$\times\{(1-is_1)pc_1-[i(s_1-\bar{s}_1)(1+c_1+i\bar{s}_1-i\bar{s}_1c_1)B^*+$$
$$+i(s_1-\bar{s}_2)(1+c_1+i\bar{s}_2-i\bar{s}_2c_1)B_1^*+p(1+is_1)]\}=b_1\zeta+\frac{b_{-1}}{\zeta}; \tag{9}$$

$$\omega(\zeta)=R\left(\zeta+\frac{c_1}{\zeta}\right),\quad \sigma_*=\sigma_x^\infty+\sigma_y^\infty+p. \tag{10}$$

The constant c_1 can be found by solving the algebraic equation

$$(1+s_1^2)(1+c_1-is_2+is_2c_1)[(1+c_1+is_1-is_1c_1)B^*-2a_{-1}]+$$
$$+(1+s_2^2)(1+c_1-is_1+is_1c_1)[(1+c_1+is_2-is_2c_1)B_1^*-2b_{-1}]=0. \tag{11}$$

Relation (11) simplifies considerably if $s_1=i\beta_1$, $s_2=i\beta_2$. In this particular case (11) reduces to

$$A_1c_1^2+A_2c_1+A_3=0;\quad A_1=(1-\beta_1)(1-\beta_2)(\sigma_y^\infty-\sigma_x^\infty);$$
$$A_2=A_1+(1+\beta_1)(1+\beta_2)(2p+\sigma_x^\infty+\sigma_y^\infty);$$
$$A_3=(1+\beta_1)(1+\beta_2)(\sigma_y^\infty-\sigma_x^\infty). \tag{12}$$

By setting $\beta_1=\beta_2=1$, in (12) one finds the value of c_1 in the case of isotropic media, namely

$$c_1=\frac{\sigma_y^\infty-\sigma_x^\infty}{-2p-\sigma_x^\infty-\sigma_y^\infty}.$$

The "equally rigid" hole contours for (10) are given by a family of similar ellipses,

$$\frac{x^2}{(1+c_1)^2}+\frac{y^2}{(1-c_1)^2}=R^2.$$

The stressed state is determined by using the formulas (3) into which the relations (8) and (9) are substituted having previously replaced ζ by $\zeta_1=\zeta_1(z_1)$ and $\zeta_2=\zeta_2(z_2)$, respectively; the latter are obtained by inverting the formulas $z_1=\omega_1(\zeta)$ and $z_2=\omega_2(\zeta)$.

One quadrant of the sought contour for $p=0, \sigma_x^\infty=0.5\,\sigma_y^\infty$, is shown in the diagram, the small plate being made of aircraft plywood with the elastic constants [2]

$$a_{11}=\frac{1}{E_x}=0.83333\frac{10^{-9}}{9.81};\quad a_{12}=-\frac{\nu_x}{E_x}=-0.5917\frac{10^{-9}}{9.81};\quad a_{16}=0;$$

$$a_{22}=\frac{1}{E_y}=1.66667\frac{10^{-9}}{9.81};\quad a_{66}=\frac{1}{G_{xy}}=14.2857\frac{10^{-9}}{9.81};\quad a_{26}=0$$

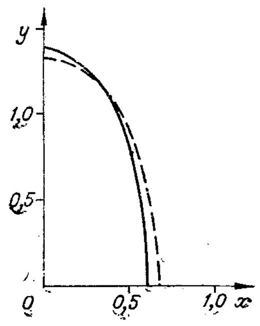

Fig. 1

and the complex-valued parameters $s_1=4.11i$; $s_2=0.343i$.

For comparison, the dashed line shows a contour quadrant for an isotropic plate.

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